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The minimal number of nodes in Chebyshev type quadrature formulas

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ABSTRACT

We study Chebyshev type quadrature formulas of degree n with respect to a weight function on $[-1,1]$, i.e. formulas

$$\frac{1}{\int_{-1}^{1} w(t)\,dt} \int_{-1}^{1} f(t)\,w(t)\,dt = \frac{1}{N}\sum_{i=1}^{N} f(x_i) + R(f)$$

with nodes $x_i \in [-1,1]$, such that $R(f)=0$ for every polynomial of degree $\leq n$. It is known that for a Jacobi weight function $w(t)=(1-t)^{\alpha}(1+t)^{\beta}$ the number of nodes has to satisfy the inequality $N \geq K_1 n^{2+2\max(\alpha,\beta)}$ for some absolute constant $K_1>0$. In this paper it is shown that for an ultraspherical weight function $w(t)=(1-t^2)^{\alpha}$ with $\alpha \geq 0$, this lower bound is of the right order, i.e. there exists a Chebyshev type quadrature formula of degree n with $N \leq K_2 n^{2+2\alpha}$ nodes. Our method of proof is based on a method of S.N. Bernstein who obtained the result in case $\alpha=0$. In general this method gives a large number of multiple nodes. It is also shown that the nodes can be chosen to be distinct.

1. INTRODUCTION

Let $w(t)$ be a normalized *weight function* on $[-1,1]$, i.e. $w(t)$ is a non-negative integrable function such that $\int_{-1}^{1} w(t)\,dt = 1$. A *quadrature formula* for the weight function $w(t)$ is a formula of the type

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{N} p_i f(x_i) + R(f), \tag{1.1}$$

where $x_i \in [-1,1]$, $i=1,\dots,N$ are the *nodes* and $p_i \in \mathbb{R}$, $i=1,\dots,N$ are the *weights* of the quadrature formula. $R(f)$ is the error that one makes when approximating the integral $\int_{-1}^{1} f(t)w(t)dt$ by the finite sum $\sum p_i f(x_i)$. The quadrature formula (1.1) has (algebraic) *degree* (at least) n if $R(f)=0$ for every polynomial f of degree $\leq n$. In particular (1.1) has degree ≥ 0 if and only if $\sum p_i = \int_{-1}^{1} w(t)dt=1$. The quadrature formula is *positive* if every weight p_i is non-negative.

A *Chebyshev type quadrature formula* is a quadrature formula of degree ≥ 0 in which all weights are equal, i.e. a formula of the type

$$(1.2) \qquad \int_{-1}^{1} f(t)w(t)dt = \frac{1}{N}\sum_{i=1}^{N} f(x_i)+R(f).$$

We call N the *size* of the formula. The formula (1.2) is called a *strict* Chebyshev type quadrature formula if the nodes x_i are distinct. See [6] for a review on Chebyshev quadrature.

Bernstein [1] proved that if a Chebyshev type quadrature formula (1.2) has degree $2n-1$ then

$$(1.3) \qquad N \geq \frac{1}{\lambda_{1,n}},$$

where $\lambda_{1,n}$ is the *Christoffel number* appearing in the n-point Gauss quadrature formula

$$\int_{-1}^{1} f(t)w(t)dt = \sum_{i=1}^{n} \lambda_{i,n} f(\xi_{i,n})+R(f),$$

with $1>\xi_{1,n}>\cdots>\xi_{n,n}>-1$ the zeros of the orthogonal polynomial of degree n with respect to $w(t)$. Actually, Bernstein considered only $w(t)\equiv$const, but his proof applies without difficulties to the general case.

For $w(t)\equiv\frac{1}{2}$ we have

$$n(n+1)\lambda_{1,n} < \lim_{s\to\infty} s(s+1)\lambda_{1,s} = \frac{1}{J_0'(j_1)^2} \approx 3.7104,$$

where j_1 is the first positive zero of the Bessel function $J_0(t)$. Monotonicity of the sequence $n(n+1)\lambda_{1,n}$ was recently shown by Korevaar [8]. Hence Bernstein's formula (1.3) implies the inequality

$$N > J_0'(j_1)^2 n(n+1) \approx 0.2695n(n+1).$$

The same method applied to the Jacobi weight $w(t)=C_{\alpha,\beta}(1-t)^{\alpha}(1+t)^{\beta}$, $\alpha,\beta\geq -\frac{1}{2}$ gives the inequality

$$(1.4) \qquad N \geq Kn^{2+2\max(\alpha,\beta)},$$

for some constant $K>0$ only depending on α and β, see [10] for the ultraspherical case and [3] for the general case.

In another paper [2] Bernstein obtained an inequality in the other direction. He proved that, for $w(t)\equiv\frac{1}{2}$ and every n, there exists a Chebyshev type quadrature formula of degree $2n-1$ with N nodes, where

$$(1.5) \qquad N \approx 4\sqrt{2}(n+1)(n+4).$$

In Bernstein's construction many nodes of the quadrature formula are the same. In fact, there are only $2n-1$ distinct ones.

This result does not seem to be generally known; a recent article [12] contains the weaker estimate $N=\mathcal{O}(n^3)$ for the minimal number of nodes in a Chebyshev type quadrature formula of degree $2n-1$.

In this paper I shall extend Bernstein's method to prove that for an ultraspherical weight function $w_\alpha(t)=C_\alpha(1-t^2)^\alpha$, $\alpha\geq 0$, the lower bound in (1.4) is of the right order. That is, the following theorem is proved.

MAIN THEOREM. *Let $w_\alpha(t)=C_\alpha(1-t^2)^\alpha$ with $\alpha\geq 0$. There is a constant $K=K_\alpha>0$, depending only on α such that, for every n, there exists a Chebyshev type quadrature formula* (1.2) *for the weight function $w_\alpha(t)$ of degree n having size $N\leq Kn^{2+2\alpha}$.*

Moreover, it is possible to use either a large number of multiple nodes: only $\approx n$ distinct nodes, or N distinct nodes: strict Chebyshev type quadrature.

Since Bernstein's paper [2] for the case $\alpha=0$ is only available in Russian, it seems appropriate to present our extension of Bernstein's method in full.

The method is based on the observation that a positive quadrature formula of degree $m-1$ with m distinct nodes can be perturbed a little to give a formula of degree $m-1$ having slightly different nodes and weights. As long as there are m distinct nodes the perturbation can be continued, and, as will be proved, also in some cases with multiple nodes. In this way one tries to obtain a quadrature formula with rational weights whose common denominator N is as small as possible. Such a quadrature formula can be viewed as a Chebyshev type quadrature formula of size N. A similar idea is used in [13].

2. A SPLITTING THEOREM

The following theorem, which may be of interest in itself, is our main tool in deriving the existence theorem for Chebyshev type quadrature. The case of one pair of multiple points was considered in [13].

THEOREM 2.1. *Let $1>y_1\geq y_2\geq\cdots\geq y_m>-1$ be m points such that*

(2.1) $\quad y_i=y_{i+1},\ y_j=y_{j+1}\Rightarrow j-i$ *is an even number.*

Let $p_i>0$, $i=1,\dots,m$. Then for $t>0$ sufficiently small, there exist points $x_i(t)$, $i=1,\dots,m$ such that

(2.2) $\quad 1>x_1(t)>x_2(t)>\cdots>x_m(t)>-1,$

(2.3) $\quad \lim_{t\downarrow 0} x_i(t)=y_i,\quad i=1,\dots,m,$

(2.4) $\quad \sum_{i=1}^{m} p_i x_i(t)^l=\sum_{i=1}^{m} p_i y_i^l,\quad t>0,\ l=0,\dots,m-1.$

So the points y_i which appear twice can be split in such a way that the sums in (2.4) remain constant.

PROOF. Condition (2.1) implies $y_i > y_{i+2}$, i.e. no three y_i's coincide. Let I_1 be the collection of indices i such that $y_{i-1} > y_i > y_{i+1}$ (single points) and let I_2 be the collection of indices i such that $y_i = y_{i+1}$ (double points). Here it is understood that $y_0 = 1$, $y_{m+1} = -1$. We introduce variables $t > 0$, u_i, $i = 1, \dots, m$ and put

$$(2.5)\qquad \begin{cases} x_i = y_i + u_i t^2, & \text{if } i \in I_1, \\ x_i = y_i + p_{i+1} u_{i+1} t + u_i t^2, & \text{if } i \in I_2, \\ x_{i+1} = y_i - p_i u_{i+1} t, & \text{if } i \in I_2. \end{cases}$$

For (2.4) to hold (with $x_i(t) = x_i$) we must have

$$(2.6)\qquad 0 = \sum_{i \in I_1} p_i (x_i^l - y_i^l) + \sum_{i \in I_2} [p_i x_i^l + p_{i+1} x_{i+1}^l - (p_i + p_{i+1}) y_i^l], \quad l = 1, \dots, m-1.$$

Using (2.5) we expand (2.6) into powers of t:

$$(2.7)\qquad \begin{cases} 0 = \sum_{i \in I_1} p_i l y_i^{l-1} u_i t^2 \\ \quad + \sum_{i \in I_2} \left[p_i l y_i^{l-1} u_i t^2 + (p_i + p_{i+1}) p_i p_{i+1} \dfrac{l(l-1)}{2} y_i^{l-2} u_{i+1}^2 t^2 \right] \\ \quad + l t^3 \Psi_l(t, u_1, \dots, u_m), \qquad l = 1, \dots, m-1, \end{cases}$$

where Ψ_l is a polynomial in the variables $t, u_1, \dots, u_m$. Division of (2.7) by lt^2 and the substitutions

$$(2.8)\qquad \begin{cases} v_i = p_i u_i, & \text{if } i \in I_1 \cup I_2, \\ v_{i+1} = (p_i + p_{i+1}) p_i p_{i+1} u_{i+1}^2 / 2, & \text{if } i \in I_2, \end{cases}$$

lead to the equations

$$(2.9)\qquad \sum_{i \in I_1 \cup I_2} y_i^{l-1} v_i + \sum_{i \in I_2} (l-1) y_i^{l-2} v_{i+1} = -t \tilde{\Psi}_l(t, v_1, \dots, v_m), \quad l = 1, \dots, m-1.$$

For $t = 0$ (2.9) is a homogeneous system of linear equations for the variables v_i, $i = 1, \dots, m$. Because of (2.8) we want to have a solution with $v_{i+1} > 0$ if $i \in I_2$. Now let $j \in I_2$ be the largest element of I_2 and fix $v_{j+1} = 1$. Then (2.9) with $t = 0$ and $v_{j+1} = 1$ gives a system of $m-1$ equations for the $m-1$ unknowns v_i, $i = 1, \dots, m$, $i \neq j+1$, which has the form

$$(2.10)\qquad V \begin{pmatrix} v_1 \\ \vdots \\ v_j \\ v_{j+2} \\ \vdots \\ v_m \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \\ y_j \\ \vdots \\ \vdots \\ (m-2) y_j^{m-3} \end{pmatrix}$$

with

$$V=\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 1 & 1 & \cdots \\ y_1 & \cdots & y_i & 1 & \cdots & y_j & y_{j+2} & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \\ y_1^{m-2} & \cdots & y_i^{m-2} & (m-2)y_i^{m-3} & \cdots & y_j^{m-2} & y_{j+2}^{m-2} & \cdots \end{bmatrix}.$$

The matrix $V=V(y_1,\dots,y_j,y_{j+2},\dots,y_m)$ is a confluent Vandermonde matrix, which has a column $[1,y_i,\dots,y_i^{m-2}]^t$ for every $i\in I_1\cup I_2$. For $i\in I_2$, $i\neq j$, there is an additional column $[0,1,y_i\dots,(m-2)y_i^{m-3}]^t$. This matrix is invertible and has determinant

$$\text{(2.11)}\qquad \det V(y_1,\dots,y_j,y_{j+2},\dots,y_m)=\prod_{i>i',i,i'\in I_1\cup I_2}(y_i-y_{i'})^{\mu_i\mu_{i'}},$$

where μ_i is the multiplicity to which y_i occurs in $(y_1,\dots,y_j,y_{j+2},\dots,y_m)$.

We compute v_{i+1} for $i\in I_2$ using Cramer's rule. Replacing the column $[0,1,y_i\dots,(m-2)y_i^{m-3}]^t$ by the right-hand side of (2.10), and interchanging the two columns $[1,y_i,\dots,y_i^{m-2}]^t$ and $[1,y_j,\dots,y_j^{m-2}]^t$, we obtain v_{i+1} as the quotient of two confluent Vandermonde determinants, which are computed according to (2.11):

$$\begin{aligned} v_{i+1}&=\frac{\det V(y_1,\dots,y_{i-1},y_j,y_{j+1},y_{i+2},\dots,y_{j-1},y_i,y_{j+2},\dots,y_m)}{\det V(y_1,\dots,y_j,y_{j+2},\dots,y_m)}\\ &=\prod_{i'\neq i,i+1,j,j+1}\frac{y_j-y_{i'}}{y_i-y_{i'}}. \end{aligned}$$

Thus $v_{i+1}>0$ if and only if $j-i$ is even. This is just the assumption (2.1).

So the equations (2.9) have for $t=0$ a unique solution in which $v_{j+1}=1$ and for that solution $v_{i+1}>0$ if $i\in I_2$. By the implicit function theorem there exists for small $t>0$, a unique solution $v_1(t),\dots,v_m(t)$ of (2.9) for which $v_{j+1}(t)=1$, $v_{i+1}(t)>0$, $i\in I_2$. Then

$$\text{(2.12)}\qquad \begin{cases} u_i(t)=\dfrac{v_i(t)}{p_i} & \text{if } i\in I_1\cup I_2,\\[2ex] u_{i+1}(t)=\left[\dfrac{2v_{i+1}(t)}{p_ip_{i+1}(p_i+p_{i+1})}\right]^{\frac12} & \text{if } i\in I_2, \end{cases}$$

satisfy the equations (2.7). Finally, putting

$$\text{(2.13)}\qquad \begin{cases} x_i(t)=y_i+u_i(t)t^2, & \text{if } i\in I_1,\\ x_i(t)=y_i+p_{i+1}u_{i+1}(t)t+u_i(t)t^2, & \text{if } i\in I_2,\\ x_{i+1}(t)=y_i-p_iu_{i+1}(t)t, & \text{if } i\in I_2, \end{cases}$$

we have (2.2), (2.3), (2.4). □

REMARK. In the proof we have found for small $t>0$, a unique solution $v_1(t), \ldots, v_m(t)$ of the equations (2.9) subject to the extra conditions $v_{j+1}(t)=1$ and $v_{i+1}(t)>0$ for $i \in I_2$. From (2.12), (2.13) we get

$$(2.14) \quad x_{j+1}(t) = y_j - \left[\frac{2p_j}{p_{j+1}(p_j+p_{j+1})}\right]^{\frac{1}{2}} t.$$

Now the uniqueness of $v_1(t), \ldots, v_m(t)$ implies that the points $x_1(t), \ldots, x_m(t)$ are uniquely determined by (2.2), (2.3), (2.4), (2.14).

COROLLARY 2.2. *Let* $1>y_1 \geq \cdots \geq y_m > -1$ *satisfy* (2.1) *and let* $p_i>0$, $i=1, \ldots, m$. *Suppose we have a symmetric situation, i.e.*

$$y_{m+1-i} = -y_i, \quad p_{m+1-i} = p_i, \qquad i=1, \ldots, m.$$

Then, for $t>0$ *sufficiently small, the points* $x_1(t), \ldots, x_m(t)$ *which satisfy* (2.2), (2.3), (2.4) *are also symmetric, i.e.*

$$x_{m+1-i}(t) = -x_i(t), \quad i=1, \ldots, m.$$

PROOF. Put

$$\tilde{x}_i(t) = -x_{m+1-i}(t), \quad i=1, \ldots, m.$$

By symmetry these points satisfy (2.2), (2.3), (2.4).

By (2.12) we have $u_{i+1}(t)>0$ if $i \in I_2$ and also $\lim_{t \downarrow 0} u_{i+1}(t)>0$. Then by (2.13) we see that for t sufficiently small, and $i \in I_2$, the function $x_i(t)$ is increasing and $x_{i+1}(t)$ is decreasing.

Hence, if $t>0$ is sufficiently small, $\tilde{x}_{j+1}(t) = x_{j+1}(s)$ for some small $s>0$. Because of the uniqueness property, see the preceding remark, we have $\tilde{x}_i(t) = x_i(s)$ for every i. Thus

$$(2.15) \quad x_{j+1}(t) = -x_{m-j}(s), \quad x_{j+1}(s) = -x_{m-j}(t).$$

Now suppose $s<t$. Then $x_{j+1}(t)<x_{j+1}(s)$ and $x_{m-j}(t)>x_{m-j}(s)$ which contradicts (2.15). Similarly, $s>t$ is impossible, so that $s=t$. This implies $x_i(t) = -x_{m+1-i}(t)$ for every i. □

3. DISTINCT NODES

A Chebyshev type quadrature formula with multiple nodes can be written as

$$(3.1) \quad \int_{-1}^{1} f(t) w(t) dt = \frac{1}{N} \sum_{i=1}^{N_0} A_i f(x_i) + R(f),$$

where the numbers A_i are positive integers and $\sum_i A_i = N$. As an application of Theorem 2.1 we show how to obtain from a formula (3.1) a Chebyshev type quadrature formula of the same degree having *distinct nodes*. The only restriction will be that there are "enough" distinct nodes in (3.1). Similar arguments are used in [13].

LEMMA 3.1. *Suppose*

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \frac{1}{N}\sum_{x\in E} A_x f(x) + R(f), \quad A_x \in \mathbb{N},$$

is a Chebyshev type quadrature formula of degree $m-1$, *where* E *is a finite subset of* $(-1,1)$ *with*

$$m-1 \le |E| < N.$$

Then there exists a Chebyshev type quadrature of degree $m-1$ *with* $|E|+1$ *distinct nodes.*

PROOF. Let x_0 be a node with $A_{x_0} \ge 2$ and let $E_0 \subset E$ be such that $x_0 \in E_0$ and $|E_0| = m-1$. We number the elements of E_0 in decreasing order, where x_0 is numbered twice. So $E_0 = \{y_1, \dots, y_m\}$ such that, for some i_0,

$$1 > y_1 > \cdots > y_{i_0} = x_0 = y_{i_0+1} > \cdots > y_m > -1.$$

Put

$$p_{i_0} = A_{x_0} - 1, \quad p_{i_0+1} = 1, \quad p_i = A_{y_i}, \qquad i \notin \{i_0, i_0+1\}.$$

The points $y_1, \dots, y_m$ satisfy the condition (2.1) of Theorem 2.1 and the numbers p_i are positive integers. According to Theorem 2.1 there exist distinct points $\tilde{y}_1, \dots, \tilde{y}_m$ such that

$$\sum_{i=1}^{m} p_i y_i^l = \sum_{i=1}^{m} p_i \tilde{y}_i^l, \quad l = 0, \dots, m-1.$$

Because of (2.3) we may assume that the points $\tilde{y}_i$ are not in $E \setminus E_0$. Now we have for every polynomial f of degree $\le m-1$

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \frac{1}{N}\sum_{x\in E} A_x f(x) = \frac{1}{N}\sum_{x\notin E_0} A_x f(x) + \frac{1}{N}\sum_{i=1}^{m} p_i f(y_i)$$

$$= \frac{1}{N}\sum_{x\notin E_0} A_x f(x) + \frac{1}{N}\sum_{i=1}^{m} p_i f(\tilde{y}_i).$$

So the points $\tilde{y}_1, \dots, \tilde{y}_m$ together with the points in $E \setminus E_0$ can be used as the nodes of a Chebyshev type quadrature formula of degree $m-1$ which has $|E|+1$ distinct nodes. □

Repeated application of Lemma 3.1 establishes the following

THEOREM 3.2. *Suppose* (3.1) *is a Chebyshev type quadrature formula of degree* $m-1$ *having* $N_0 \ge m-1$ *distinct nodes* $x_i \in (-1,1)$. *Then there exists a Chebyshev type quadrature formula of degree* $m-1$ *with* N *distinct nodes in* $(-1,1)$.

4. ADMISSIBLE WEIGHT VECTORS

Let $m \in \mathbb{N}$. Vandermonde determinants imply the fact that for every m-tuple $\boldsymbol{x} = (x_1, \dots, x_m)$ with $1 > x_1 > x_2 > \cdots > x_m > -1$, there exists a unique weight vector

$$p(x) = p = (p_1, \dots, p_m)$$

such that the quadrature formula

$$\int_{-1}^{1} f(t)\, w(t)\, dt = \sum_{i=1}^{m} p_i f(x_i) + R(f) \tag{4.1}$$

has degree $m-1$.

DEFINITION 4.1. A vector $p=(p_1, \dots, p_m)$ is *admissible* if $p_i > 0$, $i=1,\dots,m$ and p equals the weight vector $p(x)$ for some $x=(x_1,\dots,x_m)$ with $1 > x_1 > \cdots > x_m > -1$.

THEOREM 4.2. *The collection of admissible weight vectors is an open subset of the collection of all vectors $(p_1,\dots,p_m)$ satisfying $\sum_{i=1}^m p_i = 1$.*

PROOF. Let $p=p(x)$ be admissible. Inserting the monomials t^k into (4.1) we find

$$\sum_{i=1}^{m} p_i x_i^k = \int_{-1}^{1} t^k w(t)\, dt, \quad k=0,1,\dots,m-1.$$

Partial differentiation with respect to x_j gives

$$\sum_{i=1}^{m} x_i^k \frac{\partial p_i}{\partial x_j} = -k p_j x_j^{k-1}, \quad k=0,1,\dots,m-1.$$

Using matrix notation

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_m \\ x_1^2 & & & x_m^2 \\ \vdots & & & \vdots \\ x_1^{m-1} & \cdots & \cdots & x_m^{m-1} \end{pmatrix} \begin{pmatrix} \partial p_1/\partial x_1 & \partial p_1/\partial x_2 & \cdots & \partial p_1/\partial x_m \\ \partial p_2/\partial x_1 & & & \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \partial p_m/\partial x_1 & \cdots & \cdots & \partial p_m/\partial x_m \end{pmatrix}$$

$$= - \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & 0 & & \vdots \\ 0 & 2 & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & m-1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & & x_m \\ \vdots & & \vdots \\ \vdots & & \vdots \\ x_1^{m-1} & \cdots & x_m^{m-1} \end{pmatrix} \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & \ddots \\ & & & & p_m \end{pmatrix}.$$

If $p_i > 0$, $i=1,\dots,m$, then the Jacobi matrix $\partial p/\partial x$ has rank $m-1$. So in that case the map $x \mapsto p(x)$ is locally surjective onto the hyperplane $\sum_{i=1}^m p_i = 1$. This proves the theorem. □

An application of Theorem 2.1 gives

PROPOSITION 4.3. *If $p_1,\dots,p_m>0$, $1>x_1\geq\cdots\geq x_m>-1$, are such that the quadrature formula*

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{m} p_i f(x_i) + R(f)$$

has degree $m-1$ and if $x_i=x_{i+1}$, $x_j=x_{j+1}\Rightarrow j-i$ even, then $(p_1,\dots,p_m)$ is admissible.

PROOF. According to Theorem 2.1 there exist $1>\tilde{x}_1>\cdots>\tilde{x}_m>-1$ such that

$$\sum_{i=1}^{m} p_i \tilde{x}_i^k = \sum_{i=1}^{m} p_i x_i^k, \quad k=0,\dots,m-1.$$

Then the quadrature formula

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{m} p_i f(\tilde{x}_i) + R(f)$$

has degree $m-1$ and $(p_1,\dots,p_m)$ is admissible. □

As a special case of Proposition 4.3 we mention the Gauss quadrature formula. Suppose m is even, $m=2n$. Let $P_n(t)$ be the orthogonal polynomial of degree n with respect to the weight $w(t)$. Let $1>\xi_1>\xi_2>\cdots>\xi_n>-1$ be the n zeros of $P_n(t)$, numbered in decreasing order. There exist unique positive numbers $\lambda_1,\dots,\lambda_n$ (Christoffel numbers), such that

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{n} \lambda_i f(\xi_i) + R(f) \tag{4.2}$$

has degree $2n-1$. Proposition 4.3 immediately gives

COROLLARY 4.4. *If $p_1,\dots,p_m>0$ satisfy $p_{2i-1}+p_{2i}=\lambda_i$, $i=1,\dots,n$, then $(p_1,\dots,p_m)$ is admissible.*

5. CANONICAL QUADRATURE FORMULAS

In this section we survey some definitions, notations and results concerning special quadrature formulas, which will be needed later. Most of the material can be found in a more general setting in [9] and [7, Chapters II & IV].

We define

$$\varepsilon(x) = \begin{cases} 1 & \text{if } -1<x<1, \\ \frac{1}{2} & \text{if } |x|=1. \end{cases}$$

Following Karlin–Studden [7] we say that the *index* of a quadrature formula

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{N} p_i f(x_i) + R(f),$$

with $p_i>0$, $1\geq x_1>\cdots>x_N\geq -1$, is the number $\sum_{i=1}^{N} \varepsilon(x_i)$.

Let $n \in \mathbb{N}$ be fixed. We shall consider positive quadrature formulas of degree $2n-1$. A quadrature formula is called *principal* if the index is n and *canonical* if the index is n or $n+\frac{1}{2}$. For every $x \in (-1,1)$ there exists a unique canonical quadrature formula which involves the point x. We denote by $\lambda(x)$ the weight on x in the canonical quadrature formula involving x. For $x=-1$ and $x=+1$ there exists a unique principal quadrature formula (the Lobatto quadrature formula, see below) which involves both end-points. We denote by $\lambda(-1)$ and $\lambda(1)$ their respective weights in the Lobatto quadrature formula.

The canonical quadrature formulas can be classified as follows. Let $P_n(t)$ be the orthogonal polynomial of degree n with respect to $w(t)$ and let $1>\xi_1>\cdots>\xi_n>-1$ be the zeros of $P_n(t)$. Let $Q_{n-1}(t)$ be the orthogonal polynomial of degree $n-1$ with respect to the weight function $(1-t^2)w(t)$. The $n+1$ zeros of $(1-t^2)Q_{n-1}(t)$ are also numbered in decreasing order $1=\eta_0>\eta_1>\eta_2>\cdots>\eta_{n-1}>\eta_n=-1$. We assume that $P_n(1)>0$ and $Q_{n-1}(1)>0$. Also consider, for every a, the polynomial $P_n(t,a)$ defined by

$$P_n(t,a) = \begin{cases} P_n(t)-a(1-t)Q_{n-1}(t), & a \geq 0, \\ P_n(t)-a(1+t)Q_{n-1}(t), & a \leq 0. \end{cases}$$

For every $a \in \mathbb{R}$ the polynomial $P_n(t,a)$ has n simple zeros in the interval $(-1,1)$ denoted by $1>\xi_1(a)>\cdots>\xi_n(a)>-1$. If $a>0$ then $\xi_i<\xi_i(a)<\eta_{i-1}$. If $a<0$ then $\eta_i<\xi_i(a)<\xi_i$. If a increases from $-\infty$ to ∞, then $\xi_i(a)$ increases from η_i to η_{i-1}. Occasionally we write $\xi_i(\infty):=\eta_{i-1}$, $\xi_i(-\infty)=\eta_i$.

There exist two principal quadrature formulas. One of them is the Gauss quadrature formula, which has $\xi_1>\xi_2>\cdots>\xi_n$ as its nodes:

$$(5.1) \qquad \int_{-1}^{1} f(t)w(t)\,dt = \sum_{i=1}^{n} \lambda(\xi_i)f(\xi_i)+R(f).$$

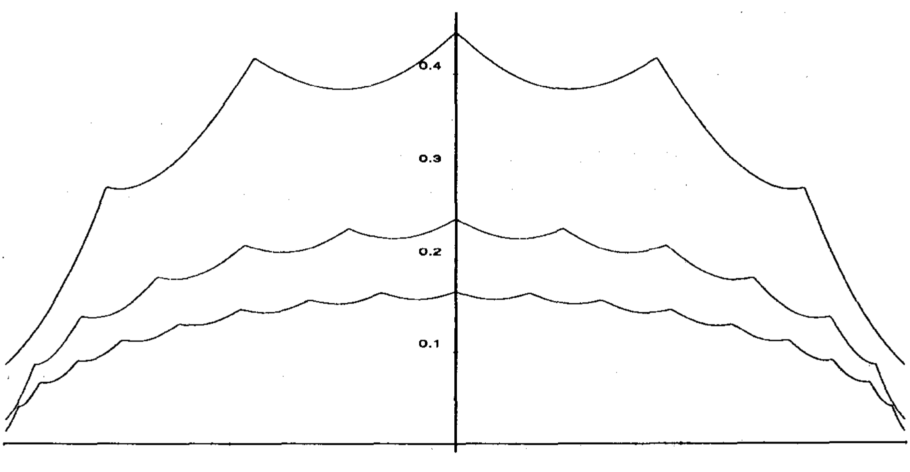

Fig. 1. The function $\lambda(x)$ for $w(t)\equiv 1/2$ and $n=3,6,9$.

The numbers $\lambda(\xi_i)$ are the Christoffel numbers. The other principal quadrature formula has the nodes $\eta_0 > \eta_1 > \cdots > \eta_n$, and is called the *Lobatto quadrature formula*:

$$(5.2) \qquad \int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=0}^{n} \lambda(\eta_i) f(\eta_i) + R(f).$$

The quadrature formulas with index $n+\frac{1}{2}$ split into two groups; one which involves -1 and one which involves $+1$. These are called *lower canonical* and *upper canonical*, respectively. For $a>0$ the points $\xi_1(a) > \cdots > \xi_n(a) > -1$ are the nodes of a lower canonical quadrature formula:

$$(5.3) \qquad \int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{n} \lambda(\xi_i(a)) f(\xi_i(a)) + \varrho_a f(-1) + R(f),$$

and every lower canonical quadrature formula is obtained in this way. For $a<0$ the points $1 > \xi_1(a) > \cdots > \xi_n(a)$ are the nodes of an upper canonical quadrature formula:

$$(5.4) \qquad \int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{n} \lambda(\xi_i(a)) f(\xi_i(a)) + \varrho_a f(1) + R(f),$$

and every upper canonical quadrature formula is obtained in this manner.

We have defined $\lambda(x)$ as the weight of x in the canonical quadrature formula involving x. The number $\lambda(x)$ can be characterized by certain extremal properties.

$\lambda(x)$ is the maximal weight which a positive quadrature formula of degree $2n-1$ can have at the point x. Moreover, the canonical quadrature formula involving x is the only positive quadrature formula of degree $2n-1$ attaining this maximum. [Note that we only consider quadrature formulas with nodes in $[-1,1]$. In this respect there is a difference with the Christoffel function as considered for example by Freud [5].]

The number $\lambda(x)$ can also be obtained as the minimal integral of a certain class of functions:

$$\lambda(x) = \min \int_{-1}^{1} f(t)\,w(t)\,dt,$$

where the minimum is taken over all polynomials $f(t)$ of degree $\leq 2n-1$ which satisfy

$$f(t) \geq 0 \text{ on } [-1,1], \quad f(x) = 1.$$

Next we introduce the functions $\pi(x)$ and $\underline{\pi}(x)$ as the total weight which the canonical quadrature formula involving x has on the interval $[x,1]$, and on the interval $(x,1]$, respectively. For $x = \pm 1$ we take the Lobatto quadrature formula. If $x = \xi_i(a)$ with $a \geq 0$ then

$$\pi(x) = \sum_{j=1}^{i} \lambda(\xi_j(a)), \quad \underline{\pi}(x) = \sum_{j=1}^{i-1} \lambda(\xi_j(a)).$$

If $x=\xi_i(a)$ with $a<0$ then

$$\pi(x) = \sum_{j=1}^{i} \lambda(\xi_j(a))+\varrho_a, \quad \underline{\pi}(x) = \sum_{j=1}^{i-1} \lambda(\xi_j(a))+\varrho_a,$$

with ϱ_a from formula (5.4). If $x=\eta_i$ then

$$\pi(x) = \sum_{j=0}^{i} \lambda(\eta_j), \quad \underline{\pi}(x) = \sum_{j=0}^{i-1} \lambda(\eta_j).$$

$\pi(x)$ is a continuous positive function on $[-1,1]$. For $x\in[-1,\xi_n]$ it assumes the value 1. On the interval $[\xi_n,1]$ it is strictly decreasing. For $x\in[\xi_1,1]$ we have the value $\pi(x)=\lambda(x)$. Similar remarks hold for $\underline{\pi}(x)$.

From the definitions it is clear that

$$\underline{\pi}(x)+\lambda(x)=\pi(x), \tag{5.5}$$

and

$$\pi(\xi_i(a)) = \underline{\pi}(\xi_{i+1}(a)). \tag{5.6}$$

The Markov–Stieltjes inequalities, cf. Freud [5, I.5], state that

$$\underline{\pi}(x) \leq \int_x^1 w(t)\,dt \leq \pi(x). \tag{5.7}$$

Let

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{N} p_i f(x_i)+R(f)$$

be a positive quadrature formula of degree $2n-1$ and let $x\in[-1,1]$. The Markov–Stieltjes inequalities also give

$$\underline{\pi}(x) \leq \sum_{j,x_j>x} p_j \leq \sum_{j,x_j\geq x} p_j \leq \pi(x). \tag{5.8}$$

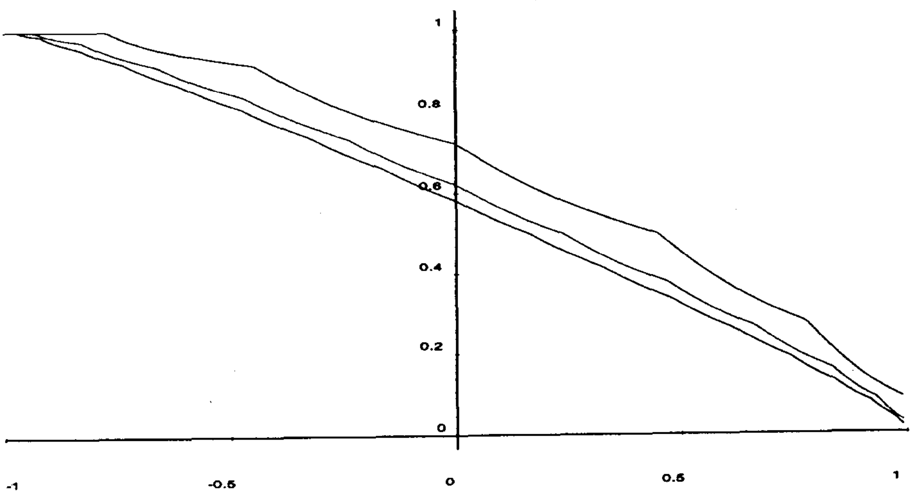

Fig. 2. The function $\pi(x)$ for $w(t)\equiv 1/2$ and $n=3,6,9$.

So $\pi(x)$ is the maximal weight which a positive quadrature formula of degree $2n-1$ can have on the interval $[x,1]$ and $\underline{\pi}(x)$ is the minimal weight of a positive quadrature formula of degree $2n-1$ on $(x,1]$. In both cases the canonical quadrature formula involving x is the only quadrature formula for which the maximum or minimum is attained.

The function $\pi(x)$ can also be characterized by an extremal property of polynomials:

$$\pi(x) = \min \int_{-1}^{1} f(t)\,w(t)\,dt, \tag{5.9}$$

where the minimum is taken over all polynomials $f(t)$ of degree $\leq 2n-1$ which satisfy

$$f(t) \geq 0 \text{ on } [-1,x], \quad f(t) \geq 1 \text{ on } [x,1]. \tag{5.10}$$

The minimizing polynomial can be described in terms of the nodes of the canonical quadrature formula involving x. For example, if $x=\xi_i(a)$, $a>0$ then $f(t)$ is determined by the properties

$$f(\xi_j(a)) = 1, \quad f'(\xi_j(a)) = 0, \qquad j=1,\dots,i-1,$$
$$f(\xi_j(a)) = 0, \quad f'(\xi_j(a)) = 0, \qquad j=i+1,\dots,n,$$
$$f(\xi_i(a)) = 1, \quad f(-1) = 0.$$

Similarly,

$$\underline{\pi}(x) = \max \int_{-1}^{1} f(t)\,w(t)\,dt,$$

where the maximum is taken over all polynomials $f(t)$ of degree $\leq 2n-1$ satisfying

$$f(t) \leq 0 \text{ on } [-1,x], \quad f(t) \leq 1 \text{ on } [x,1].$$

6. SYMMETRIC WEIGHT FUNCTION

In the remaining sections of this paper we consider only *symmetric* weight functions $w(t)$, i.e. $w(-t)=w(t)$, and we are looking for quadrature formulas

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{m} p_i f(x_i) + R(f) \tag{6.1}$$

which are *symmetric*, in the sence that $x_{m+1-i}=-x_i$, $p_{m+1-i}=p_i$. So (6.1) can be written as

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{n} p_i[f(x_i)+f(-x_i)] + R(f), \tag{6.2}$$

where as before $m=2n$.

DEFINITION 6.1. A vector $(p_1,\dots,p_n)$ with $\sum p_i=\frac{1}{2}$, $p_i>0$, $i=1,\dots,n$, is *s-admissible* if there exists a quadrature formula of the form (6.2) of degree $2n-1$ with $1>x_1>x_2>\cdots>x_{n-1}>x_n>0$.

In this section we prove a theorem which gives sufficient conditions on a vector $(p_1, \dots, p_n)$ to be s-admissible. These conditions will be stated in terms of the function $\pi(x)$ introduced in Section 5.

The results of Section 4 on admissible vectors also hold for s-admissible vectors. We shall need the following results.

PROPOSITION 6.2. *The collection of s-admissible vectors is an open subset of the hyperplane* $\sum p_i = \frac{1}{2}$.

PROOF. Same proof as Theorem 5.2. □

PROPOSITION 6.3. *If a symmetric quadrature formula of the form* (6.2) *with* $p_i > 0$,

$$(6.3) \qquad 1 > x_1 > x_2 > \cdots > x_{n-1} > 0, \quad x_{n-1} \geq x_n \geq 0,$$

has degree $2n-1$, *then* $(p_1, \dots, p_n)$ *is s-admissible.*

PROOF. Equality among the points x_i, $i = 1, \dots, m$, where $x_{m+1-i} = -x_i$, can only occur in one of the following two ways: 1) $x_n = x_{n+1} = 0$, or 2) $x_{n-1} = x_n > 0 > x_{n+1} = x_{n+2}$. In both cases Proposition 4.3 can be applied. Using Corollary 2.2 we find that the points $\tilde{x}_1, \dots, \tilde{x}_m$ in the proof of Proposition 4.3 are symmetric. Hence $(p_1, \dots, p_n)$ is s-admissible. □

We now arrive at the main existence theorem.

THEOREM 6.4. *Let* $w(t)$ *be a symmetric weight function. Suppose n even,* $n = 2l$. *Let*

$$(6.4) \qquad \infty = a_1 > a_2 > \cdots > a_{l+1} \geq 0 \geq b_1 > b_2 > \cdots > b_l \geq -\infty,$$

and $p_1, \dots, p_n > 0$, $\sum p_i = \frac{1}{2}$ *be such that*

$$(6.5) \qquad \pi(\xi_i(a_i)) \leq p_1 + p_2 + \cdots + p_{2i-1} < \pi(\xi_i(a_{i+1})), \quad i = 1, \dots, l.$$

$$(6.6) \qquad \pi(\xi_i(b_i)) \leq p_1 + p_2 + \cdots + p_{2i} < \pi(\xi_i(b_{i+1})), \quad i = 1, \dots, l-1.$$

Then $(p_1, \dots, p_n)$ *is s-admissible.*

For the proof of Theorem 6.4 we need some lemmas.

LEMMA 6.5. Let $a_1, \dots, a_{l+1}$, $b_1, \dots, b_l$, *and* $p_1, \dots, p_n > 0$ *satisfy the inequalities* (6.4), (6.5), (6.6) *and assume there exist* $1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ *such that the quadrature formula*

$$(6.7) \qquad \int_{-1}^{1} f(t)\, w(t)\, dt = \sum_{i=1}^{n} p_i [f(x_i) + f(-x_i)] + R(f),$$

has degree $2n-1$. *Then* $(p_1, \dots, p_n)$ *is s-admissible.*

PROOF. From the last inequality in (5.8) with $x=x_{2i-1}$ and the first inequality in (6.5), we obtain $\pi(\xi_i(a_i))\le\pi(x_{2i-1})$. Since π is strictly decreasing on $[0,1]$ we find

$$x_{2i-1}\le\xi_i(a_i),\quad i=1,\dots,l. \tag{6.8}$$

The first inequality in (5.8) with $x=x_{2i-1}$ combined with the second inequality in (6.6) yields $\pi(x_{2i-1})<\pi(\xi_{i-1}(b_i))$. Because of (5.6) this implies

$$\xi_i(b_i)<x_{2i-1},\quad i=2,\dots,l. \tag{6.9}$$

Similar considerations with $x=x_{2i}$ give

$$x_{2i}\le\xi_i(b_i),\quad i=1,\dots,l-1. \tag{6.10}$$

$$\xi_{i+1}(a_{i+1})<x_{2i},\quad i=1,\dots,l. \tag{6.11}$$

The inequalities (6.8), (6.9), (6.10) and (6.11) show that

$$\xi_1\ge x_2>x_3>\cdots>x_{n-2}>x_{n-1}>0.$$

In order to apply Proposition 6.3 we also need the inequalities $1>x_1>\xi_1$.

If $x_1=1$, then $p_1=\pi(1)=\lambda(1)$ by (6.5). This means that (6.7) is a positive quadrature formula of degree $2n-1$ with maximal weight at 1, and therefore is the Lobatto quadrature formula. However, this is impossible because we already have too many strict inequalities for the points x_i.

If $x_1=\xi_1$ then (6.7) would be a positive quadrature formula of degree $2n-1$ having no nodes in $(\xi_1,1]$. The only formula with this property is the Gauss quadrature formula, see e.g. [7, Lemma II.3.1], so we again have a contradiction. [For small n the preceding discussion does not hold. In that case one should argue as in Proposition 6.3 or Corollary 4.2.]

Hence the inequalities (6.3) of Proposition 6.3 are satisfied and it follows that $(p_1,\dots,p_n)$ is s-admissible.

LEMMA 6.6. *There exist $a_1^\circ,\dots,a_{l+1}^\circ$, $b_1^\circ,\dots,b_l^\circ$ and a vector $(p_1^\circ,\dots,p_n^\circ)$ which is s-admissible such that the inequalities* (6.4), (6.5), (6.6) *hold, with the strict inequalities $<$ and $>$ replaced by $\le$ and $\ge$, respectively.*

PROOF. We take $a_i^\circ=\infty$, $b_i^\circ=0$ for every i, and

$$p_{2i-1}^\circ=\pi(\eta_{i-1})-\pi(\xi_{i-1}),\quad p_{2i}^\circ=\pi(\xi_i)-\pi(\eta_{i-1}),\qquad i=1,\dots,l.$$

Here $\pi(\xi_0):=0$ for convenience. The weights p_{2i-1}°, p_{2i}°, $i=1,\dots,l$ are positive, and $\sum_{i=1}^n p_i^\circ=\pi(\xi_l)=\frac{1}{2}$. Since $p_{2i}^\circ+p_{2i-1}^\circ=\pi(\xi_i)-\pi(\xi_{i-1})=\lambda_i$, the vector $(p_1^\circ,\dots,p_n^\circ)$ is s-admissible, see Corollary 4.4. □

PROOF OF THEOREM 6.4. Suppose $a_1,\dots,a_{l+1}$, $b_1,\dots,b_l$ and $p_1,\dots,p_n$ satisfy the hypotheses of the theorem. Let $a_1^\circ,\dots,a_{l+1}^\circ$, $b_1^\circ,\dots,b_l^\circ$, $p_1^\circ,\dots,p_n^\circ$ be as in Lemma 6.6. Let $a_1(t),\dots,a_{l+1}(t)$, $b_1(t),\dots,b_l(t)$, be continuous functions of

$t \in [0,1]$ such that

$$\infty = a_1(t) > \cdots > a_{l+1}(t) \geq 0 \geq b_1(t) > \cdots > b_l(t) \geq -\infty, \quad t>0,$$

$$a_i(0) = a_i^\circ, \quad a_i(1) = a_i, \quad b_i(0) = b_i^\circ, \quad b_i(1) = b_i.$$

Then it is possible to take $p_1(t), \ldots, p_n(t) > 0$, $\sum p_i(t) = \frac{1}{2}$, continuously depending on $t \in [0,1]$, such that for every $t \in (0,1]$,

$$\pi(\xi_i(a_i(t))) \leq p_1(t) + \cdots + p_{2i-1}(t) < \pi(\xi_i(a_{i+1}(t))), \quad i = 1, \ldots, l, \tag{6.12}$$

$$\pi(\xi_i(b_i(t))) \leq p_1(t) + \cdots + p_{2i}(t) < \pi(\xi_i(b_{i+1}(t))), \quad i =, \ldots, l-1, \tag{6.13}$$

and

$$p_i(0) = p_i^\circ, \quad p_i(1) = p_i. \tag{6.14}$$

Let τ be the supremum of the collection of $t \in [0,1]$ for which the vector $(p_1(t), \ldots, p_n(t))$ is s-admissible. By Proposition 6.2 and Lemma 6.6 we have $0 < \tau \leq 1$.

We shall prove that the vector $(p_1(\tau), \ldots, p_n(\tau))$ is s-admissible. Indeed, τ is the limit of a sequence (t_j) for which $(p_1(t_j), \ldots, p_n(t_j))$ is s-admissible. Hence, for every j, there exist nodes $1 > x_1(t_j) > \cdots > x_n(t_j) > 0$ such that the quadrature formula

$$\int_{-1}^{1} f(t)\, w(t)\, dt = \sum_{i=1}^{n} p_i(t_j) [f(x_i(t_j)) + f(-x_i(t_j))] + R(f) \tag{6.15}$$

has degree $2n-1$. Choosing a subsequence if necessary we may assume that $(x_i(t_j))_j$ converges for every i, with limit x_i, say. Taking the limit $j \to \infty$ in (6.15), we see that the quadrature formula

$$\int_{-1}^{1} f(t)\, w(t)\, dt = \sum_{i=1}^{n} p_i(\tau) [f(x_i) + f(-x_i)] + R(f) \tag{6.16}$$

has degree $2n-1$. We also have $1 \geq x_1 \geq \cdots \geq x_n \geq 0$. Then because of the inequalities (6.12), (6.13) it follows by Lemma 6.5 that $(p_1(\tau), \ldots, p_n(\tau))$ is s-admissible.

If τ would be <1, then we would conclude from Proposition 6.2 that $(p_1(t), \ldots, p_n(t))$ is s-admissible for t in a neighbourhood of τ. Hence $\tau = 1$, and it follows by (6.14) that $(p_1, \ldots, p_n)$ is s-admissible. □

From Theorem 6.4 we obtain the following criterion for the existence of a quadrature formula with rational weights having a pre-assigned common denominator.

COROLLARY 6.7. *Let $\infty = a_1 > a_2 > \cdots > a_{l+1} \geq 0 \geq b_1 > \cdots > b_l \geq -\infty$, and let N be an even number such that*

$$\pi(\xi_i(a_{i+1})) - \pi(\xi_i(a_i)) \geq \frac{1}{N}, \quad i = 1, \ldots, l, \tag{6.17}$$

$$\pi(\xi_i(b_{i+1})) - \pi(\xi_i(b_i)) \geq \frac{1}{N}, \quad i = 1, \ldots, l-1. \tag{6.18}$$

Then there exist $1>x_1>\cdots>x_n>0$ *and positive integers* $A_1,\dots,A_n$*, such that the quadrature formula*

$$(6.19)\qquad \int_{-1}^{1} f(t)\,w(t)\,dt=\frac{1}{N}\sum_{i=1}^{n} A_i[f(x_i)+f(-x_i)]+R(f)$$

has degree $2n-1$.

PROOF. Because of (6.17) with $i=1$ one can find a positive integer A_1 such that $\pi(\xi_1(a_1))\le A_1/N<\pi(\xi_1(a_2))$. Then $A_1/N<\pi(\xi_1(b_1))$ and so by (6.18) with $i=1$ there is a positive integer A_2 such that $\pi(\xi_1(b_1))\le A_1/N+A_2/N<\pi(\xi_1(b_2))$. Next, it follows that $A_1/N+A_2/N<\pi(\xi_2(a_2))$, so that by (6.17) with $i=2$ there is a positive integer A_3 such that $\pi(\xi_2(a_2))\le A_1/N+A_2/N+A_3/N<\pi(\xi_2(a_3))$.

Continuing in this way we find positive integers $A_1,\dots,A_{n-1}$ such that $p_1=A_1/N,\dots,p_{n-1}=A_{n-1}/N$ satisfy the inequalities (6.5), (6.6). Then $A_n=N/2-\sum_{i=1}^{n-1}A_i$ is also a positive integer, since N is even. It follows from Theorem 6.4 that $(A_1/N,\dots,A_n/N)$ is s-admissible. □

The quadrature formula (6.19) is a Chebyshev type quadrature formula in which many nodes coincide; the node x_i has multiplicity A_i. By Theorem 3.2 one can obtain from this a Chebyshev type quadrature formula of degree $2n-1$ with N distinct nodes. Nevertheless, if the Chebyshev type quadrature formula (1.2) is optimal in a certain sense, there will be many coincident nodes, cf. [11] for the case of a constant weight function.

7. ULTRASPHERICAL WEIGHT FUNCTION

We consider in this section the normalized ultraspherical weight function

$$w_\alpha(t)=C_\alpha(1-t^2)^\alpha,\quad C_\alpha=2^{-2\alpha-1}\frac{\Gamma(2\alpha+2)}{\Gamma(\alpha+1)^2},$$

with $\alpha\ge 0$ and C_α such that $\int_{-1}^{1}w_\alpha(t)\,dt=1$. We will prove the Main Theorem as stated in the Introduction.

Since in the following discussion the number n is not fixed, we must be a bit more precise in our notation. When appropriate we append a suffix n to emphasize the dependence on n. For example we shall write $\pi_n(x)$, $\lambda_n(x)$, $\xi_{i,n}(a)$ and so on. The weight funciton $w_\alpha(t)$ is fixed, so we do not explicitly mention the dependence on α.

We start with a lemma on $\pi_n(x)$.

LEMMA 7.1. *Let* $n\in\mathbb{N}$. *The function*

$$(7.1)\qquad x\mapsto\frac{\pi_n(x)-\int_x^1 w_\alpha(t)\,dt}{(1+x)^{1+\alpha}}$$

is decreasing on $[-1,1]$.

Computer experiments show that the lemma is false if $\alpha<0$.

PROOF. Let $-1<x_0<y_0<1$. Let $f(t)$ be the polynomial of degree $\le 2n-1$ which satisfies

$$f(t)\ge 1,\ t\in[x_0,1],\quad f(t)\ge 0,\ t\in[-1,1],\quad \pi(x_0)=\int_{-1}^{1} f(t)\,w_\alpha(t)\,dt,$$

see (5.9), (5.10). Then $g(s):=f(-1+[(1+x_0)/(1+y_0)](1+s))$ is a polynomial of degree $\le 2n-1$ with

$$g(s)\ge 1,\ s\in[y_0,1],\quad g(s)\ge 0,\ s\in[-1,1],$$

so that

$$\begin{aligned}\pi_n(y_0)&\le \int_{-1}^{1} g(s)\,w_\alpha(s)\,ds\\&=\frac{1+y_0}{1+x_0}\int_{-1}^{-1+2(1+x_0)/(1+y_0)} f(t)\,w_\alpha\left(-1+\frac{1+y_0}{1+x_0}(1+t)\right)dt.\end{aligned}$$

We have

$$w_\alpha\left(-1+\frac{1+y_0}{1+x_0}(1+t)\right)\le\left(\frac{1+y_0}{1+x_0}\right)^{\alpha} w_\alpha(t), \tag{7.2}$$

and therefore

$$\begin{aligned}\pi_n(y_0)&\le\left(\frac{1+y_0}{1+x_0}\right)^{1+\alpha}\int_{-1}^{x_0} f(t)\,w_\alpha(t)\,dt\\&\quad+\frac{1+y_0}{1+x_0}\int_{x_0}^{-1+2(1+x_0)/(1+y_0)} f(t)\,w_\alpha\left(-1+\frac{1+y_0}{1+x_0}(1+t)\right)dt\\&=\left(\frac{1+y_0}{1+x_0}\right)^{1+\alpha}\pi_n(x_0)\\&\quad-\frac{1+y_0}{1+x_0}\int_{x_0}^{1} f(t)\left[\left(\frac{1+y_0}{1+x_0}\right)^{\alpha} w_\alpha(t)-w_\alpha\left(-1+\frac{1+y_0}{1+x_0}(1+t)\right)\right]dt,\end{aligned}$$

where $w_\alpha(s)\equiv 0$ if $s>1$. Using (7.2) again and the fast that $f(t)\ge 1$ on $[x_0,1]$ we obtain

$$\begin{aligned}\pi_n(y_0)&\le\left(\frac{1+y_0}{1+x_0}\right)^{1+\alpha}\pi_n(x_0)\\&\quad-\frac{1+y_0}{1+x_0}\int_{x_0}^{1}\left[\left(\frac{1+y_0}{1+x_0}\right)^{\alpha} w_\alpha(t)-w_\alpha\left(-1+\frac{1+y_0}{1+x_0}(1+t)\right)\right]dt\\&=\left(\frac{1+y_0}{1+x_0}\right)^{1+\alpha}\left(\pi_n(x_0)-\int_{x_0}^{1} w_\alpha(t)\,dt\right)\\&\quad+\frac{1+y_0}{1+x_0}\int_{x_0}^{-1+2(1+x_0)/(1+y_0)} w_\alpha\left(-1+\frac{1+y_0}{1+x_0}(1+t)\right)dt\\&=\left(\frac{1+y_0}{1+x_0}\right)^{1+\alpha}\left(\pi_n(x_0)-\int_{x_0}^{1} w_\alpha(t)\,dt\right)+\int_{y_0}^{1} w_\alpha(s)\,ds.\end{aligned}$$

Now it readily follows that (7.1) is a decreasing function. □

PROPOSITION 7.2. *Let $\alpha \geq 0$. There exist a constant $B_1 > 0$ and n_0 such that for $n \geq n_0$ and $0 \leq x < y \leq 1$,*

$$(7.3) \qquad \pi_n(x) - \pi_n(y) \geq (y-x)\frac{B_1}{n^{2\alpha}}.$$

In the following proofs the numbers $K_1, K_2, \ldots$ are positive constants which do not depend on x and n.

PROOF. We prove that whenever $\pi_n(x)$ is differentiable at $x \in [0,1]$ then

$$(7.4) \qquad -\pi_n'(x) \geq \frac{B_1}{n^{2\alpha}}.$$

From this (7.3) readily follows since π_n is a decreasing function.

We first assume $x \in [0, \xi_{1,n}]$. Differentiating the function (7.1) we obtain

$$|\pi_n'(x)| \geq w_\alpha(x) - (1+\alpha)\frac{\pi_n(x) - \int_x^1 w_\alpha(t)\,dt}{1+x}.$$

Let $x = \xi_{i,n}(a)$ and put $x^+ = \xi_{i+1,n}(a)$. Then by (5.6) and (5.7) $\pi_n(x) = \underline{\pi}_n(x^+) \leq \int_{x^+}^1 w_\alpha(t)\,dt$, so that $\pi_n(x) - \int_x^1 w_\alpha(t)\,dt \leq \int_{x^+}^x w_\alpha(t)\,dt$. From [14, Theorem 8.9.1] it follows that

$$(7.5) \qquad \theta_{i+1,n} - \theta_{i,n} \leq \frac{K_1}{n}, \quad i=0,\ldots,n, \quad \theta_{0,n} = 0, \quad \cos\theta_{i,n} = \xi_{i,n}, \quad i=1,\ldots,n.$$

From [14, Theorem 8.1.2] we have $\lim_n n\theta_{1,n} = j_{1,\alpha}$ where $j_{1,\alpha}$ is the first positive zero of the Bessel function $J_\alpha(t)$, so that

$$(7.6) \qquad \theta_{1,n} \geq \frac{K_2}{n}.$$

From (7.5) it follows that $x^- - x^+ \leq K_3/n$ and from (7.5) and (7.6) it is readily deduced that $w_\alpha(x^+) \leq K_4 w_\alpha(x)$. Thus $\lambda_n(x) \leq (K_3K_4/n)\,w_\alpha(x)$ and if n is large, i.e. $n \geq 2(1+\alpha)K_3K_4$, we have for $x \in [0, \xi_{1,n}]$,

$$|\pi_n'(x)| \geq \frac{w_\alpha(x)}{2} \geq \frac{w_\alpha(\xi_{1,n})}{2} \geq \frac{K_5}{n^{2\alpha}}.$$

For $x \in [\xi_{1,n}, 1]$ we proceed differently. Suppose $x = \xi_{1,n}(a)$, $a \geq 0$ and let

$$\pi_n(x) = \lambda_n(x) = \int_{-1}^{1} f(t)\,w_\alpha(t)\,dt$$

where $f(t)$ is the polynomial of degree $\leq 2n-1$ which has double zeros for $t = \xi_{2,n}(a), \ldots, \xi_{n,n}(a)$, a simple zero for $t=-1$ and which is normalized such that $f(x)=1$. On the interval $[\xi_{2,n}(a), 1]$ the polynomial $f(t)$ is increasing and convex. Therefore, if $y > x$,

$$f(y) > \frac{y - \xi_{2,n}(a)}{x - \xi_{2,n}(a)}.$$

For $y \in (x,1)$, $t \mapsto f(t)/f(y)$ is a polynomial of degree $\le 2n-1$ satisfying $f(t)/f(y) \ge 0$, $t \in [-1,1]$, $f(t)/f(y) \ge 1$, $t \in [y,1]$. Hence $\pi_n(y) \le \pi_n(x)/f(y)$ and it follows that

$$\pi_n(x) - \pi_n(y) \ge \pi_n(x)\left[1 - \frac{1}{f(y)}\right] \ge (y-x)\frac{\pi_n(x)}{y - \xi_{2,n}(a)} \ge (y-x)\frac{\pi_n(1)}{1-\xi_{2,n}}.$$

From (7.5) we see $(1-\xi_{2,n}) \le K_6/n^2$ so that we shall be finished if we can prove that

$$\pi_n(1) \ge \frac{K_7}{n^{2+2\alpha}}. \tag{7.7}$$

Recall that $\pi_n(1)$ is the weight in the Lobatto quadrature formula at the point 1. The ultraspherical polynomials $P_n^{(\alpha+1,\alpha+1)}(t)$ are orthogonal with respect to $(1-t^2)w_\alpha(t)$ and so the zeros of $(1-t^2)P_{n-1}^{(\alpha+1,\alpha+1)}(t)$ are the nodes of the Lobatto quadrature formula. It follows that

$$\pi_n(1) = \{2P_{n-1}^{(\alpha+1,\alpha+1)}(1)\}^{-1} \int_{-1}^{1} (1+t)P_{n-1}^{(\alpha+1,\alpha+1)}(t)\, w_\alpha(t)\, dt.$$

Here $P_{n-1}^{(\alpha+1,\alpha+1)}(1) = \binom{n+\alpha}{n-1}$. From Rodrigues' formula it follows after $n-1$ integrations by parts that

$$\begin{aligned}
&\int_{-1}^{1} (1+t)P_{n-1}^{(\alpha+1,\alpha+1)}(t)\, w_\alpha(t)\, dt \\
&\quad = C_\alpha \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \int_{-1}^{1} \left\{\frac{d^{n-1}}{dt^{n-1}}(1-t^2)^{n+\alpha}\right\}(1-t)^{-1}dt \\
&\quad = \frac{C_\alpha}{2^{n-1}} \int_{-1}^{1} (1+t)^{n+\alpha}(1-t)^\alpha dt = 2\frac{\Gamma(2\alpha+2)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+2\alpha+2)}.
\end{aligned}$$

Hence

$$\pi_n(1) = (\alpha+1)\Gamma(2\alpha+2)\frac{\Gamma(n)}{\Gamma(n+2\alpha+2)} \ge \frac{1}{2}\Gamma(2\alpha+3)\frac{1}{(n+2\alpha+1)^{2+2\alpha}}.$$

So (7.7) holds and the Proposition has been proved. □

For the orthogonal polynomials associated with the weight function $w_\alpha(t)$ we take the ultraspherical polynomials

$$P_n^{(\alpha,\alpha)}(t) = \frac{(-1)^n}{2^n n!}(1-t^2)^{-\alpha}\frac{d^n}{dt^n}(1-t^2)^{\alpha+n}.$$

For the orthogonal polynomial $Q_{n-1}(t)$ with respect to the weight $(1-t^2)w_\alpha(t)$ we now take $(d/dt)P_n^{(\alpha,\alpha)}(t)$. Thus

$$P_n^{(\alpha,\alpha)}(t,a) = \begin{cases} P_n^{(\alpha,\alpha)}(t) - a(1-t)\dfrac{d}{dt}P_n^{(\alpha,\alpha)}(t), & a \ge 0, \\[2ex] P_n^{(\alpha,\alpha)}(t) - a(1+t)\dfrac{d}{dt}P_n^{(\alpha,\alpha)}(t), & a \le 0. \end{cases}$$

PROPOSITION 7.3. *Let $a_i = 1/(i-1)$, $i = 1, \dots, l+1$ and let $b_i = -(i-1)/n^2$, $i = 1, \dots, l$. Then for certain positive constants B_2, B_3,*

$$\xi_{i,n}(a_i) - \xi_{i,n}(a_{i+1}) \geq \frac{B_2}{n^2}, \quad i = 1, \dots, l, \tag{7.8}$$

$$\xi_{i,n}(b_i) - \xi_{i,n}(b_{i+1}) \geq \frac{B_3}{n^2}, \quad i = 1, \dots, l-1. \tag{7.9}$$

PROOF. First we shall prove (7.8) with $i \geq 2$. For x in $(\xi_{i,n}, \eta_{i-1,n})$ we have

$$a(x) := \frac{P_n^{(\alpha,\alpha)}(x)}{(1-x)P_n^{(\alpha,\alpha)\prime}(x)} > 0$$

and, using the differential equation for the ultraspherical polynomial $P_n^{(\alpha,\alpha)}(t)$,

$$a'(x) = \frac{1-\alpha a}{1-x} + \frac{(1+\alpha)a}{1+x} + \frac{n(n+2\alpha+1)a^2}{1+x}.$$

For the inverse function $\xi_{i,n}(a)$ we have

$$\begin{aligned} \xi_{i,n}'(a) &= \left[\frac{1-\alpha a}{1-\xi_{i,n}(a)} + \frac{(1+\alpha)a}{1+\xi_{i,n}(a)} + \frac{n(n+2\alpha+1)a^2}{1+\xi_{i,n}(a)}\right]^{-1} \\ &\geq \left[\frac{1}{1-\xi_{i,n}(a)} + \frac{(1+\alpha)a}{1+\xi_{i,n}(a)} + \frac{n(n+2\alpha+1)a^2}{1+\xi_{i,n}(a)}\right]^{-1} \\ &\geq \left[\frac{1}{1-\eta_{i-1,n}} + (1+\alpha)a + n(n+2\alpha+1)a^2\right]^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} &\xi_{i,n}(a_i) - \xi_{i,n}(a_{i+1}) \\ &\quad \geq (a_i - a_{i+1})\left[\frac{1}{1-\eta_{i-1,n}} + (1+\alpha)a_i + n(n+2\alpha+1)a_i^2\right]^{-1}. \end{aligned}$$

Since for some constant K_1, (cf. [14, Section 6.21]), $1-\eta_{i-1,n} \geq K_1((i-1)^2/(n+\alpha+1)^2)$, and $(1+\alpha)a_i + n(n+2\alpha+1)a_i^2 \leq (n+\alpha+1)^2 a_i^2$, we obtain

$$\begin{aligned} \xi_{i,n}(a_i) - \xi_{i,n}(a_{i+1}) &\geq \left(\frac{1}{i-1} - \frac{1}{i}\right)\left[\frac{(n+\alpha+1)^2}{K_1(i-1)^2} + \frac{(n+\alpha+1)^2}{(i-1)^2}\right]^{-1} \\ &\geq \frac{K_1}{2(1+K_1)(n+\alpha+1)^2}. \end{aligned}$$

This proves (7.8) for $i \geq 2$.

For $i = 1$ we proceed differently. From the definition of $P_n^{(\alpha,\alpha)}(t, a)$ we see that

$$1 - \xi_{1,n}(1) = \frac{P_n^{(\alpha,\alpha)}(\xi_{1,n}(1))}{P_n^{(\alpha,\alpha)\prime}(\xi_{1,n}(1))} = \frac{(\xi_{1,n}(1) - \xi_{1,n})P_n^{(\alpha,\alpha)\prime}(\tau)}{P_n^{(\alpha,\alpha)\prime}(\xi_{1,n}(1))}$$

where τ is some number in $(\xi_{1,n}, \xi_{1,n}(1))$. Since $P_n^{(\alpha,\alpha)\prime}(t)$ is increasing on $(\xi_{1,n}, 1)$

we find

$$1-\xi_{1,n}(1) \geq \frac{(\xi_{1,n}(1)-\xi_{1,n})P_n^{(\alpha,\alpha)\prime}(\xi_{1,n})}{P_n^{(\alpha,\alpha)\prime}(1)}.$$

From [14, Theorem 8.9.1] we see $P_n^{(\alpha,\alpha)\prime}(\xi_{1,n}) \approx n^{\alpha+2}$, which means that the quotient of the two expressions remains between two positive bounds which do not depend on n. From the differential equation it follows that

$$P_n^{(\alpha,\alpha)\prime}(1) = \frac{n(n+2\alpha+1)}{2(\alpha+1)} P_n^{(\alpha,\alpha)}(1) = \frac{n(n+2\alpha+1)}{2(\alpha+1)} \binom{n+\alpha}{n} \approx n^2 \cdot n^{\alpha}.$$

Hence $1-\xi_{1,n}(1) \geq (\xi_{1,n}(1)-\xi_{1,n})/K_2$, and so $1-\xi_{1,n}(1) \geq (1-\xi_{1,n})/(K_2+1) \geq K_3/n^2$. Thus (7.8) holds also for $i=1$.

Next we prove (7.9). Let

$$b(x) := \frac{P_n^{(\alpha,\alpha)}(x)}{(1+x)P_n^{(\alpha,\alpha)\prime}(x)}.$$

Then

$$b'(x) = \frac{1+\alpha b}{1+x} - \frac{(1+\alpha)b}{1-x} + \frac{n(n+2\alpha+1)b^2}{1-x}.$$

On the interval $(\eta_{i,n}, \xi_{i,n})$ $b(x)$ is negative and the inverse function is $\xi_{i,n}(b)$, $b \in (-\infty, 0)$. Hence

$$\begin{aligned} \xi_{i,n}'(b) &= \left[\frac{1+\alpha b}{1+\xi_{i,n}(b)} - \frac{(1+\alpha)b}{1-\xi_{i,n}(b)} + \frac{n(n+2\alpha+1)b^2}{1-\xi_{i,n}(b)}\right]^{-1} \\ &\geq \left[1+\frac{-(1+\alpha)b+n(n+2\alpha+1)b^2}{1-\xi_{i,n}}\right]^{-1}. \end{aligned}$$

We have, for some $K_4>0$, not depending on i and n, $1-\xi_{i,n} \geq K_4(i^2/n^2)$ and, for every $b \in (b_{i+1}, b_i)$,

$$\begin{aligned} -(1+\alpha)b+n(n+2\alpha+1)b^2 &\leq -(1+\alpha)b_{i+1}+n(n+2\alpha+1)b_{i+1}^2 \\ &\leq K_5\frac{i^2}{n^2}, \end{aligned}$$

for some K_5. Then

$$\xi_{i,n}(b_i)-\xi_{i,n}(b_{i+1}) \geq (b_i-b_{i+1})\left[1+\frac{K_4}{K_5}\right]^{-1} = \frac{B_3}{n^2}. \qquad \square$$

PROOF OF MAIN THEOREM. Using the numbers $a_1, \dots, a_{l+1}$, $b_1, \dots, b_l$ from Proposition 7.3, we find from (7.3) and (7.8)-(7.9)

$$\pi_n(\xi_{i,n}(a_{i+1})) - \pi_n(\xi_{i,n}(a_i)) \geq \frac{B_1 B_2}{n^{2+2\alpha}}, \quad i=1,\dots,l,$$

$$\pi_n(\xi_{i,n}(b_{i+1})) - \pi_n(\xi_{i,n}(b_i)) \geq \frac{B_1 B_3}{n^{2+2\alpha}}, \quad i=1,\dots,l-1.$$

So, in view of Corollary 6.7, if N is even and

$$N \geq \frac{n^{2+2\alpha}}{B_1 \max(B_2, B_3)},$$

then there exists a Chebyshev type quadrature formula of the type (6.19) having size N.

This formula has $2n$ distinct nodes. By Theorem 3.2 there exists a Chebyshev type quadrature formula with N distinct nodes. □

REMARKS

1 The condition $\alpha \geq 0$ in the Main Theorem seems essential for our method. Of course for $\alpha = -1/2$ the Gauss quadrature formula itself has equal weights, so the Main Theorem also holds for $\alpha = -1/2$.

2 Extension to non-symmetric weight functions is possible. For Jacobi weight functions $C_{\alpha,\beta}(1-t)^\alpha(1+t)^\beta$ is is possible to show $N \leq Kn^{2+2\max(\alpha,\beta)}$ in case $\alpha, \beta \geq 0$. Details will be published elsewhere.

3 The Main Theorem has important consequences for Chebyshev type quadrature formulas in higher dimensions, see [11]. A particularly interesting case is the d-dimensional unit sphere $S^d = \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \mid \sum_{i=0}^d x_i^2 = 1\}$. A strict Chebyshev type quadrature formula on S^d is also called a spherical design, see [4]. Using the Main Theorem it can be shown that there exist strict Chebyshev type quadrature formulas on S^d of degree n having size $\mathcal{O}(n^{d(d+1)/2})$ as $n \to \infty$, as conjectured in [12]. It is not known if this is the right order. For the usual sphere S^2 one has the inequalities

$$K_1 n^2 \leq N \leq K_2 n^3$$

for the minimal size N of a Chebyshev type quadrature formula of degree n. It would be very interesting to know the precise order.

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